OCR
Oxford Cambridge and RSA

Thursday 21 October 2021 – Afternoon

AS Level Further Mathematics B (MEI)

Y414/01 Numerical Methods

Time allowed: 1 hour 15 minutes

9048733867

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

1 The table shows 3 values of x and the associated values of $f(x)$.

x	-1	1	2
$f(x)$	0.7	-3.56	-4.94

(a) Use Lagrange's method to construct the interpolating polynomial of degree 2 for the values in the table, giving your answer in the form

ax^2+bx+c

where a, b and c are constants to be determined. **[4]**

(b) Explain why it is not possible to use Newton's forward difference interpolation formula for these values. **[1]**

2 The table shows 2 values of x and the associated values of $f(x)$. The values of x are exact and the values of $f(x)$ have been rounded to 3 decimal places.

x	29.5	29.7
$f(x)$	0.492	0.495

(a) Determine the maximum possible range of values of $\frac{29.7-29.5}{f(29.7)-f(29.5)}$. **[4]**

(b) Explain why your answer to part **(a)** is so large. **[1]**

(c) Amir states that the maximum error in using the rounded values of $f(x)$ to estimate $\frac{29.7-29.5}{f(29.7)-f(29.5)}$ is the same as the maximum error in using the rounded values of $f(x)$ to estimate $\frac{29.7}{f(29.7)}-\frac{29.5}{f(29.5)}$.
Without doing any more calculations, explain whether or not he is correct. **[1]**

3 The table shows 3 values of x and the associated values of $f(x)$.

x	2	2.05	2.1
$f(x)$	1.386294	1.471572	1.558068

(a) Use the forward difference method to find two estimates of $f'(2)$. **[3]**

(b) Explain why it is not possible to use the central difference method in this case. **[1]**

(c) Explain why it is generally preferable to use the central difference method instead of the forward difference method. **[1]**

Turn over for the next question

4 **Fig. 4.1** shows the graph of $y = \ln x - 3\sqrt{x} + 4$.

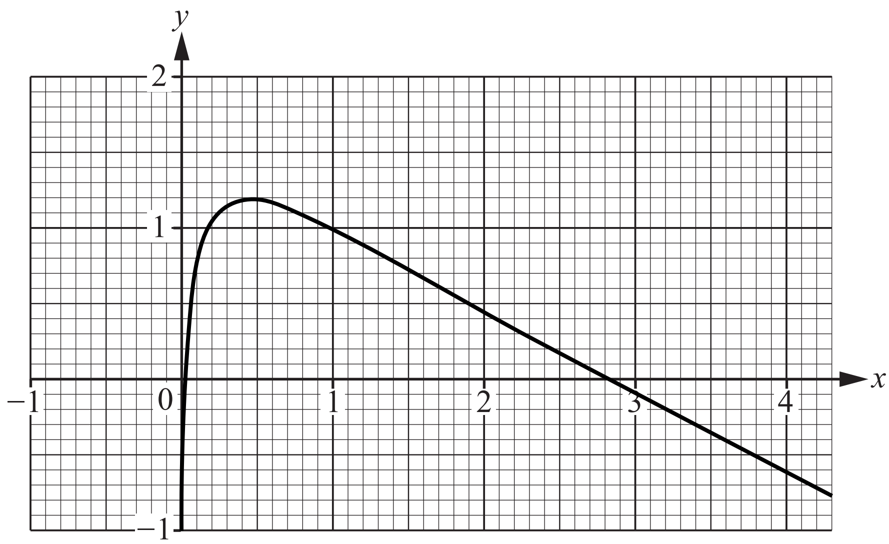

Fig. 4.1

The equation $f(x) = 0$ where $f(x) = \ln x - 3\sqrt{x} + 4$ has two roots, α and β, where $\alpha > \beta$.

The Newton-Raphson method is to be used to solve the equation.

(a) On the copy of **Fig. 4.1** in the Printed Answer Booklet, illustrate how the Newton-Raphson method finds x_1 with a starting value of $x_0 = 1$. **[1]**

The spreadsheet output in **Fig. 4.2** shows some iterates, together with some further analysis, when the Newton-Raphson method is used with a starting value of $x_0 = 0.5$.

r	x_r	difference	ratio
0	0.5		
1	10.27192	9.771918	
2	1.408102	−8.86382	−0.90707
3	2.820512	1.412411	−0.15935
4	2.817921	−0.00259	−0.00183

Fig. 4.2

(b) Determine whether $\alpha = 2.817921$ correct to **6** decimal places. **[2]**

(c) Explain what the entries in the ratio column in **Fig. 4.2** tell you about the order of convergence of this sequence of estimates. **[2]**

Fig. 4.3 shows the spreadsheet output when the Newton-Raphson method is used with a starting value of $x_0 = 0.1$.

	I	J
4	r	x_r
5	0	0.1
6	1	−0.0424
7	2	#NUM!

Fig. 4.3

(d) Explain why #NUM! is displayed in cell J7. **[2]**

The method of false position is used to find β. Some spreadsheet output showing the method being applied is shown in **Fig. 4.4**.

	N	O	P	Q	R	S
3	a	$f(a)$	b	$f(b)$	x_{new}	$f(x_{new})$
4	0.01	−0.90517	0.1	0.74873	0.05926	0.44384
5	0.01	−0.90517	0.05926	0.44384	0.04305	0.23216
6	0.01	−0.90517	0.04305	0.23216	0.0363	0.11256
7	0.01	−0.90517	0.0363	0.11256	0.03339	0.05242
8	0.01	−0.90517	0.03339	0.05242	0.03211	0.02393
9	0.01	−0.90517	0.03211	0.02393	0.03154	0.01082
10	0.01	−0.90517	0.03154	0.01082	0.03129	0.00487
11	0.01	−0.90517	0.03129	0.00487	0.03118	0.00219
12	0.01	−0.90517	0.03118	0.00219	0.03112	0.00098
13						

Fig. 4.4

(e) Determine the values which would be displayed in row 13 of the output, giving your answers correct to **5** decimal places where appropriate. **[4]**

(f) State the value of β as accurately as you can, justifying the precision quoted. **[1]**

(g) The formula in cell P9 is =IF(S8>0,R8,P8) .

Give a similar formula for cell N9. **[1]**

5 A student is trying to find an accurate approximation to $\int_1^3 f(x)\,dx$ using the trapezium rule, the midpoint rule and Simpson's rule. Some of the spreadsheet output she has generated is shown below. Some of the values are missing.

	A	B	C	D	E	F
1	n	T_n	M_n	S_{2n}	difference	ratio
2	1	2.888888889	2			
3	2	2.444444444	2.162175402	2.256265083		
4	4	2.303309923	2.224831272	2.250990823	−0.005274	
5	8	2.264070598		2.250517186	−0.000474	0.0898
6	16	2.253905539	2.248772298	2.250483378	−3.38E-05	0.07138
7	32	2.251338918	2.250052312	2.250481181	−2.2E-06	0.06499

(a) Use the information in the output to determine the values in cells

- D2,
- E3,
- F4,
- C5. **[5]**

(b) Use extrapolation to determine the value of $\int_1^3 f(x)\,dx$ as accurately as you can, justifying the precision quoted. **[5]**

BLANK PAGE

Turn over for the next question

6 James is investigating relative error. He uses $E^n = 2.718^n$ to approximate e^n, and calculates the relative error for various values of n. Some of his results are shown in the spreadsheet output in **Fig. 6.1**.

	C	D	E	F	G
3	n	1	2	3	4
4	e^n	2.7182818	7.3890561	20.0855369	54.5981500
5	E^n	2.718	7.387524	20.0792902	54.5755109
6	Rel. error	−0.0001037	−0.0002073	−0.0003110	−0.0004147
7	$-0.0001037n$	−0.0001037	−0.0002074	−0.0003110	−0.0004147

Fig. 6.1

(a) Give a suitable formula for cell D6. **[1]**

Let y be the relative error when E^n is used to approximate e^n, and x be the relative error when E^1 is used to approximate e^1.

(b) Write down a formula for y in terms of x which could be used to model the relationship between y, x and n. **[1]**

(c) Determine whether the model holds for $n = 5$. **[3]**

James tests the model again, this time using $F^n = 2.7^n$ to approximate e^n. Some of his results are shown in the spreadsheet output in **Fig. 6.2**.

	C	D	E	F	G
10	n	1	2	3	4
11	e^n	2.7182818	7.3890561	20.08553692	54.59815003
12	F^n	2.7	7.29	19.683	53.1441
13	Rel. error	−0.0067255	−0.0134058	−0.02004113	−0.02663186
14	$-0.0067255n$	−0.0067255	−0.0134510	−0.02017653	−0.02690204

Fig. 6.2

(d) Identify a limitation of the model found in part **(b)**. **[1]**

James carries out some further analysis, which is shown in the spreadsheet output in **Fig. 6.3**.

	C	D	E	F	G
17	n	1	2	3	4
18	$1/n$	1	0.5	0.333333333	0.25
19	$e^{\frac{1}{n}}$	2.71828183	1.6487213	1.395612425	1.284025417
20	$E^{\frac{1}{n}}$	2.718	1.6486358	1.395564192	1.283992134
21	Rel. error	−0.0001037	−5.184E-05	−3.4561E-05	−2.5921E-05
22		−0.0001037	−5.184E-05	−3.456E-05	−2.592E-05

Fig. 6.3

(e) Write −2.592E-05 in standard mathematical notation. **[1]**

(f) The formula in cell E22 is =E18*D21 .

This formula has been copied across row 22 so that equivalent formulae are in cells F22 and G22.

Explain the purpose of the $ symbols. **[1]**

(g) Suggest how the model identified in part **(b)** could be adapted to give a formula for z, the relative error when $E^{\frac{1}{n}}$ is used to approximate $e^{\frac{1}{n}}$, in terms of x and n. **[1]**

7 The equation $f(x) = 0$ where $f(x) = 0.9^x \times x^2 - x - 1$ has a root α, such that $\alpha \approx 1.87$.

Sundip is considering using the iterative formula $x_{n+1} = g(x_n) = \sqrt{\dfrac{x_n + 1}{0.9^{x_n}}}$ to find α to a greater precision.

(a) Use the central difference method with $h = 0.005$ to determine an estimate of $g'(1.87)$. **[2]**

(b) Explain how your answer to part **(a)** would influence Sundip's decision. **[1]**

Sundip decides to use the iterative formula given above with $x_0 = 2$. Her spreadsheet output, together with some further analysis, is shown.

	N	O	P	Q
10	r	x_r	difference	ratio
11	0	2		
12	1	1.9245009	−0.0754991	
13	2	1.8925878	−0.0319131	0.4227
14	3	1.8790715	−0.0135163	0.4235
15	4	1.8733418	−0.0057297	0.4239
16	5	1.8709120	−0.0024298	0.4241
17	6	1.8698814	−0.0010306	0.4241
18	7	1.8694442	−0.0004372	0.4242
19	8	1.8692588	−0.0001854	0.4242
20	9	1.8691801	−7.8660E-05	0.4242

(c) Write down suitable cell formulae for

- cell P12,
- cell Q13. **[2]**

(d) Explain what the entries in column Q tell you about the order of convergence of the estimates in column O. **[2]**

(e) **Without** doing any more calculations, state the value of α as accurately as you can, justifying the precision quoted. **[2]**

(f) Use the relaxed iteration $x_{n+1} = (1 - \lambda)x_n + \lambda g(x_n)$, with $\lambda = 1.736$ and $x_0 = 2$ to determine the value of α correct to **8** decimal places. **[3]**

END OF QUESTION PAPER

BLANK PAGE